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On a Class of Operators on Hilbert Space with Applications to Factorization and Systems Theory*

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A class of operators is defined in a Hilbert resolution space setting that offers a new perspective on problems of causal invertibility, special factorization, and the theory of quadratic cost optimization problems for dynamical systems. The major results include an extension of the special factorization to a class of noncompact operators and the definition of an abstract state space. These results are then used to obtain an optimal feedback solution to an abstract linear regular–quadratic cost problem.

INTRODUCTION

This paper is concerned with the investigation of a new class of operators in a Hilbert space setting that affords a new perspective on some important problems in the theory of causal invertibility, causal factorization, and the theory of dynamical systems defined on Hilbert resolution spaces (cf. [1–3]). The basic utility of this class is derived from the introduction of simple measure-theoretic notions into the resolution space theory. A brief outline of the paper follows.

In Section 1 the above class L is defined by the property of "domination by a measure" and its rudimentary properties are established. Examples of both compact and noncompact operators are given to indicate the nontriviality of the class.

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In Section 2 a Hilbert resolution space is constructed and identified with L. This leads to a new interpretation of certain transformators, i.e., operators on the space of operators, including the important transformator of triangular truncation (i.e., "taking the causal part of").

These results are then used to obtain causal invertibility and causal factorization theorems in Section 3. The measure-theoretic tools enable an especially easy proof of causal invertibility in L very similar to the standard argument used in showing that Volterra integral operators are quasinilpotent. The causal factorization theorem, although probably not as general as the Gohberg–Krein [1] result for compact operators, encompasses a large class of noncompact operators that figure prominently in the study of infinite dimensional systems.

The foregoing results are then used in Section 4 to develop a workable state space theory for an appropriate class of operators. In Section 5 this theory is used to prove that the optimal control law for the deterministic linear regular–quadratic cost problem can be represented in state feedback form. In a forthcoming paper, it will further be shown that this state space theory is sufficient to establish general formulations and solutions to problems of estimation and stochastic control.

1. Definitions and Preliminary Results

For any Banach space X, $|x|$ will denote the norm of an element $x \in X$; $B(X, Y)$ will denote the Banach space of all bounded linear maps from X into another Banach space Y, and $B(X) = B(X, X)$. We will denote Ω as a closed interval on the real line, and $a = \inf \Omega$, $b = \sup \Omega$. In the case that either a or b is infinite, we adopt for convenience the following notations: $[a, \tau] \equiv (-\infty, \tau]$ if $a = -\infty$, $\tau \in \Omega$; $[\tau, b] \equiv [\tau, \infty)$ if $b = \infty$, $\tau \in \Omega$; $[a, b]$ $(-\infty, \infty)$ if $a = -\infty$, $b = \infty$. Thus in all cases $\Omega = [a, b]$. We will denote Σ as the class of Borel subsets of Ω; S will denote the set of probability measures on Σ, and for $\mu \in S$ we define $\Omega_\mu = (\Omega, \Sigma, \mu)$. λ will always denote Lebesgue measure on R.

Let H be a (complex) Hilbert space and $E: \Sigma \to B(H)$ a resolution of the identity, i.e.,

(i) $E(\phi) = 0$, $E(\Omega) = I$,

(ii) $E(\omega)$ is a self-adjoint projection for each $\omega \in \Sigma$,

(iii) $E(\omega_1 \cap \omega_2) = E(\omega_1)\, E(\omega_2)$,

(iv) $\omega_1 \cap \omega_2 = \phi$ implies $E(\omega_1 \cup \omega_2) = E(\omega_1) + E(\omega_2)$,

(v) for each x, $y \in H$, the set function $E_{x,y}$: $\Sigma \to R$ defined by $E_{x,y}(\omega) = \langle E(\omega)\, x, y \rangle$ is a complex measure on Σ.

The pair (H, E) is called a Hilbert resolution space. E induces a time structure on H as follows: For $\tau \in \Omega$ define the projection $P^\tau \equiv E([a, \tau])$ and note that $P^b = I$. We shall always assume $P^a = 0$ and that P^τ is strongly continuous in τ. If $x \in H$, then $P^\tau x$ can be interpreted as the projection of x onto the "past" before time τ. Similarly $P_\tau x \equiv (I - P^\tau)\, x$ is the projection of x onto the "future" after time τ.

Given two Hilbert resolution spaces (H_1, E_1) and (H_2, E_2), a map $T \in B(H_1, H_2)$ is said to be *memoryless* if $E_2(\omega)\, T = TE_1(\omega)$ for all $\omega \in \Sigma$, *causal* if $P_2{}^\tau TP_1{}^\tau = P_2{}^\tau T$, and *anticausal* if $P_2{}^\tau TP_1{}^\tau = TP_1^\tau$, for all $\tau \in [a, b]$.

Let (H_1, E_1) and (H_2, E_2) be two Hilbert resolution spaces. An element $T \in B(H_1, H_2)$ is said to be *dominated by a measure* $\mu \in S$ (written $T < \mu$), if there exists $\gamma \geqslant 0$ such that $|E_2(\omega)\, T| \leqslant \gamma \sqrt{\mu(\omega)}$ for all $\omega \in \Sigma$. We denote $L_{12}^\mu = \{T \in B(H_1, H_2) \colon T < \mu\}$. This class of operators will be the focus of the paper.

The subscripts on P^τ and $E(\omega)$ will be suppressed when no ambiguity arises.

PROPOSITION 1.1. *L_{12}^μ is a subspace of $B(H_1, H_2)$, and for any $T \in L_{12}^\mu$ and $A \in B(H_1)$, we have $TA \in L_{12}^\mu$.*

Proof. Let T_1, $T_2 \in L_{12}^\mu$. Then $|E(\omega)(T_1 + T_2)| \leqslant |E(\omega)\, T_1| + |E(\omega)\, T_2| \leqslant (\gamma_1 + \gamma_2) \sqrt{\mu(\omega)}$, where $|E(\omega)\, T_i| < \gamma_i \sqrt{u(\omega)}$. Thus $T_1 + T_2 \in L_{12}^\mu$. Also for $\alpha \in C$, $T \in L_{12}^\mu$, $|E(\omega)\, \alpha T| = |\alpha|\, |E(\omega)\, T| \leqslant |\alpha|\, \gamma \sqrt{\mu(\omega)}$, and L_{12}^μ is a subspace of $B(H_1, H_2)$. Now let $A \in B(H_1)$. Then $|E(\omega)\, TA| \leqslant |E(\omega)\, T|\, |A| \leqslant \gamma\, |A| \sqrt{\mu(\omega)}$, and $TA \in L_{12}^\mu$. ∎

In particular, when the resolution spaces H_1 and H_2 coincide, $L^\mu \equiv L_{11}^\mu$ is a right ideal in $B(H_1)$.

We will also have occasion to deal with the space of operators $L_{12}^{*\mu} = \{T \in B(H_1, H_2) \colon \text{for some } \gamma,\ |TE_1(\omega)| \leqslant \gamma \sqrt{\mu(\omega)} \text{ for all } \omega \in \Sigma\}$. Clearly when $(H_1, E_1) = (H_2, E_2)$, $T \in L_{12}^\mu$ if and only if $T^* \in L_{12}^{*\mu}$.

For $T \in L_{12}^\mu$ define

$$|T|_\mu = \inf\{\gamma \colon |E(\omega)\, T| \leqslant \gamma \sqrt{\mu(\omega)} \text{ for all } \omega \in \Sigma\}.$$

We now show that L_{12}^μ is a Banach space with respect to $|\cdot|_\mu$.

THEOREM 1.2. *$|\cdot|_\mu$ defines a norm on L_{12}^μ and L_{12}^μ is complete with respect to this norm.*

Proof. The inequality $|E(\omega)(T_1 + T_2)| \leqslant |E(\omega)\, T_1| + |E(\omega)\, T_2|$ implies that $|T_1 + T_2|_\mu \leqslant |T_1|_\mu + |T_2|_\mu$. It is also evident that for $\alpha \in C$, $|\alpha T|_\mu = |\alpha|\, |T|_\mu$. Now by definition of $|\cdot|_\mu$, $|T| = |E(\Omega)\, T| \leqslant |T|_\mu$. Thus $|T|_\mu = 0$ if and only if $T = 0$, and it is verified that $|\cdot|_\mu$ defines a norm on L_{12}^μ. Now let $\{T_n\}$

be Cauchy with respect to $|\cdot|_\mu$. Since $|T| \leqslant |T|_\mu$, $\{T_n\}$ must also be Cauchy in $B(H_1, H_2)$. Hence there exists $T_0 \in B(H_1, H_2)$ such that $|T_n - T_0| \to 0$. We claim $T_0 \in L_{12}^\mu$. Let $\gamma_0 = \sup_n |T_n|_\mu$ ($\gamma_0 < \infty$ since $\{T_n\}$ is Cauchy). Then

$$\begin{aligned} |E(\omega)\, T_0| &\leqslant |E(\omega)(T_n - T_0)| + |E(\omega)\, T_n| \\ &\leqslant |T_n - T_0| + \gamma_0 \sqrt{\mu(\omega)}. \end{aligned}$$

Letting $n \to \infty$ it follows that $|T_0|_\mu \leqslant \gamma_0$. ∎

Remark. We can similarly show that $L_{12}^{*\mu}$ is also a Banach space with the norm defined as $|T|_{\mu^*} = \inf\{\gamma : |TE(\omega)| \leqslant \gamma \sqrt{\mu(\omega)}$ for all $\omega \in \Sigma\}$.

PROPOSITION 1.3. *Let $L_{12} = \bigcup_{\mu \in S} L_{12}^\mu$. Then L_{12} is a subspace of $B(H_1, H_2)$.*

Proof. Trivially, if $\alpha \in C$ and $T \in L_{12}$, then $\alpha T \in L_{12}$. Now let T_1, $T_2 \in L_{12}$ with $|T_i|_{\mu_i} = \gamma_i$. Take $\mu = 1/2(\mu_1 + \mu_2)$. Then $\mu \in S$ and $|T_i|_\mu \leqslant \sqrt{2}\, \gamma_i$. Thus $T_1 + T_2 \in L_{12}^\mu \subset L_{12}$. ∎

EXAMPLE 1.4. Let H_0 denote a Hilbert space, let $[a, b] = [0, 1]$, and take $H = L^2([0, 1], H_0)$ (with respect to Lebesgue measure λ) with the resolution of the identity E,

$$\begin{aligned} E(\omega)\, x : t &\to x(t), && t \in \omega, \\ &\to 0, && t \notin \omega. \end{aligned}$$

This resolution of the identity will be called the "truncation resolution." Let $S(t)$ be a strongly continuous semigroup on H_0 and consider the operator $T \in B(H)$ defined by

$$Tx : t \to \int_0^t S(t - \sigma)\, x(\sigma)\, d\sigma.$$

Then for a Borel subset $\omega \subset [0, 1]$,

$$|E(\omega)\, Tx|^2 = \int_\omega \left| \int_0^t S(t - \sigma)\, x(\sigma)\, d\sigma \right|^2 dt.$$

Since S is strongly continuous, there exists $M < \infty$ such that $\sup |S(t)| \leqslant M$. Hence

$$\begin{aligned} |E(\omega)\, Tx|^2 &\leqslant M^2 \int_\omega \left[\int_0^t |x(\sigma)|\, d\sigma \right]^2 dt \\ &\leqslant M^2\, |x|^2\, \lambda(\omega), \end{aligned}$$

and

$$|E(\omega)\, T| \leqslant M\sqrt{\lambda(\omega)}, \qquad \text{i.e., } T < \lambda.$$

Note that T is not necessarily compact if H_0 is infinite dimensional.

EXAMPLE 1.5. Let (H_1, E_1) and (H_2, E_2) be separable Hilbert resolution spaces, and suppose $T \in B(H_1, H_2)$ is Hilbert–Schmidt. We claim that $T < \mu$, where $\mu(\omega) = |E(\omega)\, T|^2_{HS}$. (For convenience we assume $|T|_{HS} = 1$.) To see this let $\{\omega_i\}_{i=1}^{\infty} \subset \Sigma$ be pairwise disjoint, and set $\omega = \bigcup \omega_i$. Then for a complete orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of H_1 we have

$$\mu(\omega) = \sum_j \langle E(\omega)\, Te_j, E(\omega)\, Te_j \rangle$$

$$= \sum_j \sum_{i,l} \langle E(\omega_i)\, Te_j, E(\omega_l)\, Te_j \rangle.$$

Since $E(\omega_i)\, E(\omega_l) = 0$ for $i \neq l$,

$$\mu(\omega) = \sum_j \sum_i \langle E(\omega_i)\, Te_j, E(\omega_i)\, Te_j \rangle$$

$$= \sum_i \sum_j \langle E(\omega_i)\, Te_j, E(\omega_i)\, Te_j \rangle$$

$$= \sum_i |E(\omega_i)\, T|^2_{HS}$$

$$= \sum_i \mu(\omega_i).$$

Thus μ is countably additive. Now since $|E(\omega)\, T| \leqslant |E(\omega)\, T|_{HS}$, it follows that $T < \mu$.

In the example above if we further assume the map $t \to \langle P^t x, x \rangle$ to be absolutely continuous for each $x \in H_2$, it follows that $\lambda(\omega) = 0$ implies $|E(\omega)\, x|^2 = 0$. Hence, in this case $\mu \ll \lambda$. In a separable Hilbert space this assumption can always be made after an appropriate reparameterization (see [1, p. 220] for details). Operators bounded by a probability measure absolutely continuous with respect to Lebesgue measure will play an important role in Section 3. Note also that since T^* is Hilbert–Schmidt, there exists a probability μ such that $T \in L^{\mu} \cap L^{*\mu}$ (if $T < \mu_1$, $T^* < \mu_2$, take $\mu = (\mu_1 + \mu_2)/2$).

For a Hilbert–Schmidt map T on $L^2([a, b], R^n)$,

$$Tx : t \to \int_{[a,b]} T(t, s)\, x(s)\, ds,$$

we can take

$$\mu(\omega)=\int_{\omega} g(t)\,dt,$$

where

$$g(t)=\int_{[a,b]} |T(t,s)|^2\,ds.$$

This has an immediate extension to operator valued kernels when R^n is replaced by an arbitrary Hilbert space.

2. A Resolution Space Characterization of L^{μ}_{12}

In this section we give an interpretation of the Banach space L^{μ}_{12} that will figure prominently in applications.

Fix $\mu\in S$ and let (H_1,E_1) and (H_2,E_2) be two Hilbert resolution spaces. Let $\mathscr{H}^{\mu}$ denote the space $L^2(\Omega_\mu;H_1)$ and let E' denote the truncation resolution of the identity on $\mathscr{H}^{\mu}$:

$$\begin{aligned} E'(\omega)\,x: t &\to x(t), \qquad && t\in\omega,\\ &\to 0, && t\notin\omega. \end{aligned}$$

Further, denote the subspace of simple functions in $\mathscr{H}^{\mu}$ by $\mathscr{H}^{\mu}_0$.

Now let $x\in\mathscr{H}^{\mu}_0$, where $x(t)=\sum\chi(\omega_i)(t)\;x_i$, with $x_i\in H_1$. For $T\in L^{\mu}_{12}$ define

$$F_\mu(T)\,x=\sum E_2(\omega_i)\,Tx_i.$$

Noting that $\mu(\omega_i)=0$ implies $E_2(\omega_i)\,T=0$, it is evident that $F_\mu(T)$ is well defined on $\mathscr{H}^{\mu}_0$. Furthermore, it is clear that the mapping $F_\mu: T\to F_\mu(T)$ is linear.

Proposition 2.1. *The following hold*:

(i) *For each $T\in L^{\mu}_{12}$, $F_\mu(T)$ has a unique extension $\mathscr{F}_\mu(T)$ such that $\mathscr{F}_\mu(T)\in B(\mathscr{H}^{\mu},H_2)$,*

(ii) *$\mathscr{F}_\mu\in B(L^{\mu}_{12},B(\mathscr{H}^{\mu},H_2))$.*

Proof. (i) Let $x\in\mathscr{H}^{\mu}_0$ with $x(t)=\sum\chi(\omega_i)(t)\,x_i$. Then,

$$\begin{aligned} |F_\mu(T)\,x|^2 &= |\textstyle\sum E_2(\omega_i)\,T x_i|^2=\sum |E_2(\omega_i)\,Tx_i|^2\leqslant\sum|E_2(\omega_i)\,T|^2\,|x_i|^2\\ &\leqslant |T|^2_\mu\textstyle\sum|x_i|^2\,\mu(\omega_i)=|T|^2_\mu\,|x|^2_{\mathscr{H}}, \end{aligned}$$

where $|\cdot|_{\mathscr{H}}$ denotes the norm in $\mathscr{H}^u$. Thus $F_u(T)$ is bounded on $\mathscr{H}_0^u$. Since $\mathscr{H}_0^u$ is dense in $\mathscr{H}^u$, $F_u(T)$ has a continuous extension $\mathscr{F}_u(T) \in B(\mathscr{H}^u, H_2)$.

(ii) As was just shown, $|\mathscr{F}_\mu(T)| \leqslant |T|_\mu$. Thus $\mathscr{F}_\mu \in B(L_{12}^u, B(\mathscr{H}^u, H_2))$ with $|\mathscr{F}_\mu| \leqslant 1$. ∎

The major result of this section is the identification of the subspace of memoryless operators in $B(\mathscr{H}^u, H_2)$ with L_{12}^u. This identification is made precise in

THEOREM 2.2. *Let* $\mathscr{F}_\mu$ *be defined as in Proposition 2.1. Let* $\mathscr{M}^u = \{M \in B(\mathscr{H}^u, H_2): E_2(\omega) M = ME'(\omega)$ *for all* $\omega \in \Sigma\}$ *(equipped with the subspace topology). Then* $\mathscr{F}_\mu : L_{12}^u \to \mathscr{M}^u$ *is an isometric isomorphism.*

Proof. We first show that $R(\mathscr{F}^u) = \mathscr{M}^u$. Take $x \in \mathscr{H}_0^u$. Then $E_2(\omega)\mathscr{F}_\mu(T)x = E_2(\omega)\sum E_2(\omega_i)Tx_i$ (where $x = \sum \chi(\omega_i)x_i$). Hence $E_2(\omega)\mathscr{F}_\mu(T)x = \sum E_2(\omega \cap \omega_i)Tx_i$. Now, $E'(\omega)x = \sum \chi(\omega \cap \omega_i)x_i$ implies that $\mathscr{F}_\mu(T)E'(\omega)x = \sum E_2(\omega \cap \omega_i)Tx_i$. Thus $E_2(\omega)\mathscr{F}_\mu(T)/\mathscr{H}_0^u = \mathscr{F}_\mu(T)E'(\omega)/\mathscr{H}_0^u$. By continuity and the fact that $\mathscr{H}_0^u$ is dense in $\mathscr{H}^u$, it follows that $R(\mathscr{F}_\mu) \subset \mathscr{M}^u$. For the opposite inclusion let $M \in \mathscr{M}^u$ and let $i: H_1 \to \mathscr{H}^u$ denote the inclusion map, i.e., $(ix)(t) = x$ for all t. Then, $|E_2(\omega)Mi| = |ME'(\omega)i| \leqslant |M||E'(\omega)i|$. But,

$$|E'(\omega)i|^2 = \sup_{\substack{x \in H_1 \\ |x| = 1}} \int_\omega |x|^2 \, d\mu = \mu(\omega).$$

Therefore $M \cdot i \in L_{12}^u$ with $|M \cdot i|_\mu \leqslant |M|$. Now take $x \in \mathscr{H}_0^u$, $x = \sum x(\omega_j)x_j$. Then $\mathscr{F}_\mu(M \cdot i)x = \sum E_2(\omega_j)(M \cdot i)x_j = \sum ME'(\omega_j)ix_j = \sum ME'(\omega_j)x = \sum E_2(\omega_j)Mx = Mx$. So by continuity, $\mathscr{F}_\mu(M \cdot i) = M$, and $R(\mathscr{F}_\mu) = \mathscr{M}^u$. Next we show $\mathscr{F}_\mu$ is an isometry. Define $\mathscr{G} \in B(M^u, L_{12}^u)$ by $\mathscr{G}(M) = M \cdot i$. Then by the argument above $|\mathscr{G}| \leqslant 1$ and $\mathscr{F}_\mu \mathscr{G} = I$. Also for $T \in L_{12}^u$ we have $(\mathscr{G}\mathscr{F}_\mu)(T) = \mathscr{F}_\mu(T) \cdot i = T$ (this last equality is easily verified). Therefore $\mathscr{G} = \mathscr{F}_\mu^{-1}$. Now since $|\mathscr{F}_\mu|, |\mathscr{F}_\mu^{-1}| \leqslant 1$,

$$|x| = |\mathscr{F}_\mu^{-1}(\mathscr{F}_\mu x)| \leqslant |\mathscr{F}_\mu^{-1}||\mathscr{F}_\mu x| \leqslant |\mathscr{F}_\mu x| \leqslant |x|,$$

and

$$|\mathscr{F}_\mu x| = |x|. \quad ∎$$

EXAMPLE 2.3. Let $A(t) \in B(H_1)$ for each $t \in [a, b]$ with $A(t)$ strongly continuous and $\sup_t |A(t)| < \infty$. Define the map $\mathscr{A}$ on $\mathscr{H}^u$ by $(\mathscr{A}x)(t) = A(t)x(t)$. It is easily seen that $\mathscr{A} \in B(\mathscr{H}^u)$ and is memoryless. Now let $T \in L_{12}^u$. Then $\mathscr{F}_\mu(T)\mathscr{A} \in B(\mathscr{H}^u, H_2)$ is also memoryless since both $\mathscr{F}_\mu(T)$ and $\mathscr{A}$ are so. By Theorem 2.2 there exists $S \in L_{12}^u$ such that $\mathscr{F}_\mu(S) = \mathscr{F}_\mu(T)\mathscr{A}$. In the case that $A(t) \equiv A$, we can take $S = TA$.

We now introduce the transformator of triangular truncation. Define G^+, $G^- \in B(H_1, \mathscr{H}^\mu)$ by $(G^+x)(t) = P^t x$ and $(G^-x)(t) = P_t x$. Let $i \in B(H_1, \mathscr{H}^\mu)$ be the injection map, i.e., $(ix)(t) \equiv x$. Then clearly $i = G^+ + G^-$. Also, for each $T \in L^\mu_{12}$ it is evident that $T = \mathscr{F}_\mu(T) \cdot i$. From these considerations, each $T \in L^\mu_{12}$ has the additive decomposition

$$T = [T]_C + [T]_A,$$

where $[T]_C = \mathscr{F}_\mu(T)\, G^+$ and $[T]_A = \mathscr{F}_\mu(T)\, G^-$. Since $\mathscr{F}_\mu(T)$ is memoryless and G^+ is causal, it follows that $[T]_C$ is causal. Similarly, $[T]_A$ is anticausal.

THEOREM 2.4. *Both $[\cdot]_C$ and $[\cdot]_A$ are bounded projections on L^μ_{12} and every $T \in L^{12}_\mu$ has the unique decomposition $T = [T]_C + [T]_A$.*

Proof. Let $T \in L^\mu_{12}$. Then

$$\begin{aligned} \sup_{|x|=1} |E(\omega)\mathscr{F}_\mu(T)\, G^+x| &= \sup_{|x|=1} |\mathscr{F}_\mu(T)\, E'(\omega)\, G^+x| \\ &\leqslant \sup_{|x|=1} |\mathscr{F}_\mu(T)|\,|E'(\omega)\, G^+x| \\ &\leqslant \sup_{|x|=1} |T|_\mu\, |E'(\omega)\, G^+x|. \end{aligned}$$

But $|E'(\omega)\, G^+x| = [\int_\omega |P^t x|^2\, d\mu]^{1/2} \leqslant |x| \sqrt{\mu(\omega)}$. Hence $|\mathscr{F}_\mu(T)\, G^+|_\mu \leqslant |T|_\mu$ and $[\cdot]_C \in B(L^\mu_{12})$. Now assume $T \in L^\mu_{12}$ is causal and let $x \in H_1$. Then G^-x can be approximated in $\mathscr{H}^\mu$ by simple functions of the form $x_s(t) = \sum \chi[t_i, t_{i+1})(t)\, P_{t_{i+1}} x$.
But

$$\mathscr{F}_\mu(T)\, x_s = \sum E([t_i, t_{i+1}))\, TP_{t_i} x = 0$$

since T is causal. Passing to the limit it follows that $\mathscr{F}_\mu(T)\, G^-x = 0$ for all $x \in H_1$. Therefore $T = \mathscr{F}_\mu(T)\, G^+$. Combining this with the result that $[\cdot]_C \in B(L^\mu_{12})$, we have $[\cdot]_C^2 = [\cdot]_C$. Similarly we can show that $[\cdot]_A$ is also a bounded projection. The decomposition follows from $i = G^+ + G^-$, and uniqueness follows from $[[T]_A]_C = 0$. ∎

3. CAUSAL INVERTIBILITY AND FACTORIZATION IN L^μ

In this section we shall establish some basic invertibility and factorization results which are fundamental for applications in control and estimation.

Let (H, E) be a Hilbert resolution space and as before set $L^\mu = L^\mu_{11}$, where

$H_1 = H$. Let $A(t)$ be a $B(H)$ valued function on $[a, b]$. For $T \in B(H)$ define the projection integrals

$$\int dPTA(t), \qquad \int A(t)\, TdP$$

as limits (when they exist) of Riemann sums of the forms

$$\sum_{i=0}^{n-1} (P^{t_{i+1}} - P^{t_i})\, TA(t_i') \qquad \text{and} \qquad \sum_{i=0}^{n-1} A(t_i')\, T(P^{t_{i+1}} - P^{t_i}),$$

respectively, where $a = t_0 < t_1 < \cdots < t_n = b$ and $t_i' \in [t_i, t_{i+1}]$. When $A(t)$ is strongly continuous and uniformly bounded, and $T \in L^\mu$, we have previously shown that $\int dPTA(t)$ exists as a strong limit of the Riemann sums, and in fact converges to $\mathscr{F}_\mu(T)\mathscr{A} \cdot i$ (cf. Example 2.3). The following stronger assertions will be used in the sequel.

LEMMA 3.1. *Let $T \in L^\mu$ with $\mu \ll \lambda$. Then the integrals*

$$\int dPTP^t, \qquad \int P^t T^*\, dP,$$

converge uniformly in the operator topology.

Proof. Clearly $\int dPTP^t$ converges uniformly if and only if $\int P^t T^* dP$ does. Let $\varepsilon > 0$ and choose a partition π of $[a, b]$, $a = t_0 < t_1 < \cdots < t_n = b$, such that $\mu([t_i, t_{i+1}]) < \varepsilon$. Then for any refinement $\pi' > \pi$, $\pi' = \{t_{i_j}\}$

$$\left| \sum_i \sum_j E([t_{i_j}, t_{i_{j+1}}])\, TP^{t'_{i_j}} x - \sum_i E([t_i, t_{i+1}])\, TP^{t'_i} x \right|^2$$

$$= \left| \sum_i \sum_j E([t_{i_j}, t_{i_{j+1}}]\, T(P^{t'_{i_j}} - P^{t'_i})\, x \right|^2$$

$$\leqslant \sum_i \sum_j |E([t_{i_j}, t_{i_{j+1}})\, T|^2\, |(P^{t'_{i_j}} - P^{t'_i})\, x|^2$$

$$\leqslant |T|_\mu^2 \sum_i \sum_j \mu([t_{i_j}, t_{i_{j+1}}])|\, (P^{t'_{i_j}} - P^{t'_i})\, x|^2$$

$$\leqslant |T|_\mu^2 \sum_i \mu([t_i, t_{i+1}])\, |(P^{t_{i+1}} - P^{t_i})\, x|^2$$

$$\leqslant \varepsilon\, |T|_\mu^2\, |x|^2$$

and the lemma is proved. ∎

Remark. Since $\int dPTP^t = [T]_C \in L^\mu$, it follows that $\int P^t T^* \, dP = [T]_C^* \in L^{*\mu}$.

LEMMA 3.2. *Let $T \in L^\mu$ and let $A(t) \in B(H)$ for each $t \in [a, b]$, with $A(t)$ continuous with respect to the operator topology and $\sup_t |A(t)| < \infty$. Then the integrals*

$$\int dPTA(t), \qquad \int A^*(t)\, T^* dP$$

converge uniformly in the operator topology.

Proof. Again one integral converges if and only if the other does. Let $\sup_t |A(t)| < M$ and choose $\varepsilon > 0$. Let π be a partition of $[a, b]$, $a = t_0 < t_1 < \cdots < t_n = b$, such that $\mu([t_0, t_1])$, $\mu([t_{n-1}, t_n]) < \varepsilon$ and for $t_1 \leqslant t'$, $t'' \leqslant t_{n-1}$ with $|t' - t''| \leqslant \max_{i=1,n-2}(t_{i+1} - t_i)$ the inequality $|A(t') - A(t'')| < \varepsilon$ holds. Such partitions exist since $\mu([a, b]) = 1$ and $A(t)$ is uniformly continuous on compact subsets of $[a, b]$. Let $\pi' > \pi$, $\pi' = \{t_{i_j}\}$. Then

$$\begin{aligned}
&\Big|\sum_i \sum_j E([t_{i_j}, t_{i_{j+1}}])\, TA(t'_{i_j}) - \sum_i E([t_i, t_{i+1}])\, TA(t'_i)\Big|^2 \\
&\quad = \Big|\sum_i \sum_j E([t_{i_j}, t_{i_{j+1}}])\, T(A(t'_{i_j}) - A(t'_i))\Big|^2 \\
&\quad \leqslant \sum_i \sum_j |E([t_{i_j}, t_{i_{j+1}}])\, T|^2\, |A(t'_{i_j}) - A(t'_i)|^2 \\
&\quad \leqslant \sum_j |E([t_{0_j}, t_{0_{j+1}}])\, T|^2\, |A(t'_{0_j}) - A(t'_0)|^2 \\
&\qquad + \sum_{i=1}^{n-2} \sum_j |E([t_{i_j}, t_{i_{j+1}}])\, T|^2\, |A(t'_{i_j}) - A(t'_i)|^2 \\
&\qquad + \sum_j |E([t_{n-1_j}, t_{n-1_{j+1}}])\, T|^2\, |A(t'_{n-1_j}) - A(t'_{n-1})|^2 \\
&\quad \leqslant (2M\,|T|_\mu^2 + 1)\, \varepsilon^2. \quad \blacksquare
\end{aligned}$$

Remark. We have shown that $\int dPTA(t) = \mathcal{F}_\mu(T)\, \mathscr{A} \cdot i$, where $\mathscr{A} \in B(\mathscr{H}^\mu)$ is defined as $(\mathscr{A}x)(t) = A(t)\, x(t)$. Thus $\int dPTA(t) \in L^\mu$. It also follows that $\int A^*(t)\, T^* dP \in L^{*\mu}$.

THEOREM 3.3. *Suppose T is causal and $T \in L^\mu$ with $\mu \ll \lambda$. Then*

(i) *T is quasinilpotent,*

(ii) *$(I + T)^{-1} - I \in L^\mu$ and is also causal.*

Proof. (i) Let $\pi = \{t_i\}$ be a given partition of $[a, b]$, $a = t_0 < t_1 < \cdots < t_n = b$. Set $\omega_i = [t_{i-1}, t_i]$. Since T is causal we have $T = \sum_{i=1}^{n} E(\omega_i) TP^{t_i}$. It follows that

$$T^k = \sum_{i1=1}^{n} E(\omega_{i1}) TP^{t_{i1}} \sum_{i2=1}^{n} E(\omega_{i2}) TP^{t_{i2}} \cdots \sum_{ik=1}^{n} E(\omega_{ik}) TP^{t_{ik}}. \tag{1}$$

Now

$$\begin{aligned}
&\left| \sum_{i1=1}^{n} E(\omega_{i1}) TP^{t_{i1}} \cdots \sum_{ik=1}^{n} E(\omega_{ik}) TP^{t_{ik}} \right|^2 \\
&\quad = \left| \sum_{i1=1}^{n} E(\omega_{i1}) T \sum_{i2 \leqslant i1} E(\omega_{i2}) T \cdots \sum_{ik \leqslant ik-1} E(\omega_{ik}) TP^{t_{ik}} \right|^2 \\
&\quad \leqslant |T|_\mu^{2k} \sum_{i1=1}^{n} \mu(\omega_{i1}) \sum_{i2 \leqslant i1} \mu(\omega_{i2}) \cdots \sum_{ik \leqslant ik-1} \mu(\omega_{ik}).
\end{aligned} \tag{2}$$

The inequality $\mu \ll \lambda$ implies there exists $g \in L^1([a, b])$ with $g(t) \geqslant 0$ such that $\mu(\omega) = \int_\omega g(t)\, dt$ for all $\omega \in \Sigma$. It then follows from (1) and (2) (letting $|\pi| \to 0$) that

$$|T^k|^2 \leqslant |T|_\mu^{2k} \int_{[a,b]} g(t_1)\, dt_1 \int_a^{t_1} g(t_2)\, dt_2 \cdots \int_a^{t_{k-1}} g(t_k)\, dt_k \leqslant |T|_\mu^{2k}/k!. \tag{3}$$

Thus $|T^k|^{1/k} \to 0$ as $k \to \infty$.

(ii) Let $R = (I + T)^{-1} - I$. By Proposition 1.1, $R \in L^\mu$ since $R = -T - TR$. It is evident that for each $t \in [a, b]$, P^tTP^t and P^tT are also quasinilpotent. From this and the causality of T we obtain the identities $P^tRP^t = (I + P^tTP^t)^{-1} - I$ and $P^tR = (I + P^tT)^{-1} - I$. But since $P^tTP^t = P^tT$, it follows that $P^tRP^t = P^tR$ for each t, hence R is causal. ∎

COROLLARY 3.4. *Theorem* 3.3 *is also valid under the hypotheses $T \in L^\mu$ is anticausal, or $T \in L^{*\mu}$ is causal or anticausal.*

Proof. If $T \in L^\mu$ is anticausal the proof of the theorem remains valid with the modification in the limits of integration in the iterated integral (3) to

$$\int_{[a,b]} g(t_1)\, dt_1 \int_{t_1}^{b} g(t_2)\, dt_2 \cdots \int_{t_{k-1}}^{b} g(t_k)\, dt_k.$$

Now if $T \in L^{*\mu}$ is causal (anticausal), then $T^* \in L^\mu$ is anticausal (causal). Thus $|T^n|^{1/n} = |T^{*n}|^{1/n} \to 0$ as $n \to \infty$. ∎

The fact that a causal $T \in L^\mu$ with $\mu \ll \lambda$ is quasinilpotent actually follows from Lemma 3.1 via a result of DeSantis and Porter [4]. Indeed, the hypotheses of Theorem 3.3 imply that T is uniformly strictly causal, hence $(I+T)^{-1}-I$ is also uniformly strictly causal [4]. We include our proof here to demonstrate the utility of the concept "$T<\mu$" as a sort of bridge between resolution space theory and functional analysis.

Our next result concerns factorization of operators of the form $(I-T)$ into its Volterra factors,

$$(I-T)=(I+X^{+})(I+X^{-}), \tag{4}$$

where X^{+} and X^{-} are respectively causal and anticausal. We shall say that $(I-T)$ admits a *special factorization* if in addition to (4), $(I+X^{+})^{-1}$ and $(I+X^{-})^{-1}$ both exist and are respectively causal and anticausal. The fundamental results on factorization for compact T are due to Gohberg and Krein [1]. In particular, these authors show that if T is in the Macaev ideal and $(I-P^tTP^t)$ is invertible for each $t\in[a,b]$, then $(I-T)$ admits a unique special factorization. Larson [5] has recently shown the existence of arbitrarily small (in norm) compact perturbations of the identity for which no special factorization exists.

We now prove our major result on special factorization. For compact operators it is probably not as general as the Gohberg–Krein results, but it does encompass a wide class of noncompact operators that figure prominently in control and optimization problems.

THEOREM 3.5. *Let $T\in L^{\mu}\cap L^{*\mu}$ with $\mu\ll\lambda$, and assume $(I-T)$ is invertible. Then $(I-T)$ has a unique special factorization*

$$(I-T)=(I+X^{+})(I+X^{-}),$$

with X^{+}, $X^{-}\in L^{\mu}\cap L^{*\mu}$ if and only if $(I-P^tTP^t)$ is invertible for each $t\in[a,b]$.

Proof (Necessity). Suppose $(I-T)=(I+X^{+})(I+X^{-})$ with X^{+}, $X^{-}\in L^{\mu}\cap L^{*\mu}$. Then

$$(I-P^tTP^t)=(I+P^tX^{+}P^t)(I+P^tX^{-}P^t).$$

Since $P^tE(\omega)=E(\omega)P^t$, it follows that $P^tX^{+}P^t$, $P^tX^{-}P^t\in L^{\mu}\cap L^{*\mu}$ for each $t\in[a,b]$. Thus $(I+P^tX^{+}P^t)$ and $(I+P^tX^{-}P^t)$ are invertible by Theorem 3.3 and Corollary 3.4. Consequently, $(I-P^tTP^t)$ is also invertible.

(Sufficiency). We first claim that the map $t\to(I-P^tTP^t)^{-1}=I+R(t)$ (where $R(t)=(I-P^tTP^t)^{-1}-I$) is continuous. Choose $\varepsilon>0$. By absolute continuity there exists $\delta>0$ such that $|t_1-t_2|<\delta$ implies $\mu([t_1,t_2])<\varepsilon^2$. Then

$$|P^{t_2}TP^{t_2} - P^{t_1}TP^{t_1}| \leqslant |P^{t_2}TP^{t_2} - P^{t_1}TP^{t_2}| + |P^{t_1}TP^{t_2} - P^{t_1}TP^{t_1}|,$$
$$\leqslant |E([t_1, t_2])\,T| + |TE([t_1, t_2])|,$$
$$\leqslant (|T|_\mu + |T|_{\mu^*})\,\varepsilon.$$

Thus $t \to P^tTP^t$ is continuous. Since the taking of inverses is a continuous operation, it follows that $R(t)$ is also continuous, and the claim is proved. Recalling that $(I - T)$ is invertible, using the finiteness of μ and continuity of the inverse, it also follows that $\sup_t |R(t)| < \infty$. Now consider the partition π of $[a, b]$, $a = t_0 < t_1 < \cdots < t_n = b$, and the associated operator X_π,

$$X_\pi = \sum (I - P^{t_j}TP^{t_j})^{-1} P^{t_j}T(P^{t_{j+1}} - P^{t_j}),$$
$$= \sum R(t_j)\,T(P^{t_{j+1}} - P^{t_j}) + \sum P^{t_j}T(P^{t_{j+1}} - P^{t_j}).$$

Since $R(t)$ is continuous and bounded, the first sum on the right above converges uniformly by Lemma 3.2, while the second sum converges uniformly by Lemma 3.1. Thus $X_\pi \to \int R(t)T\,dP + \int P^tT\,dP = \tilde{X}^-$ uniformly. Now $(\tilde{X}^-)^* = \mathscr{F}_\mu(T^*)\,\mathscr{R}G^+$, where $\mathscr{R} \in B(\mathscr{H}^\mu, \mathscr{H}^\mu)$ is given by $(\mathscr{R}x)(t) = (P^t + R^*(t))\,x(t)$. Hence $(\tilde{X}^-)^* \in L^\mu$ and is causal. Taking adjoints again it follows that $\tilde{X}^- \in L^{*\mu}$ and is anticausal. Now we follow along the lines of the proof in [1]. For a partition π and $A \in B(H)$, associate the sum

$$S_\pi(A) = \sum_{j=0}^{n-1} P^{t_j}A(P^{t_{j+1}} - P^{t_j}).$$

Then

$$S_\pi(TX_\pi) = \sum_{j=0}^{n-1} P^{t_j}T \sum_{k=0}^{n-1} (I - P^{t_k}TP^{t_k})^{-1} P^{t_k}T(P^{t_{k+1}} - P^{t_k})(P^{t_{j+1}} - P^{t_j})$$
$$= \sum_{j=0}^{n-1} P^{t_j}TP^{t_j}(I - P^{t_j}TP^{t_j})^{-1} P^{t_j}T(P^{t_{j+1}} - P^{t_j}).$$

Also

$$P^{t_j}TP^{t_j}(I - P^{t_j}TP^{t_j})^{-1} = (I - P^{t_j}TP^{t_j})^{-1} - I.$$

So, $S_\pi(TX_\pi) = X_\pi - S_\pi(T)$. Or equivalently, $S_\pi(T(I + X_\pi)) = X_\pi$. Similarly, for any partition π and $A \in B(H)$, associate the sum

$$S'_\pi(A) = \sum_j (P^{t_{j+1}} - P^{t_j})\,AP^{t_j}.$$

Since $(S_\pi + S'_\pi)(A) = A$,

$$-X_\pi + T(I + X_\pi) = S'_\pi(T(I + X_\pi)). \tag{5}$$

Thus

$$(I-T)(I+X_\pi)=I-S'_\pi(T(I+X_\pi)). \tag{6}$$

Since $X_\pi \to \tilde{X}^-$ uniformly, $S'_\pi(T(I+X_\pi))$ converges uniformly to some $X^+ \in B(H)$. We claim $X^+ = \mathscr{F}_\mu(T(I+\tilde{X}^-))\,G^+ \in L^\mu$. Let $x \in H$. Then $\sum E([t_i,t_{i+1}])\,T(I+\tilde{X}^-)\,P^{t_i}x \to \mathscr{F}_\mu(T(I+\tilde{X}^-))\,G^+x$ as $\mu([t_i,t_{i+1}])\to 0$. By definition

$$S'_\pi(T(I+X_\pi))\,x=\sum E([t_i,t_{i+1}])\,T(I+X_\pi)\,P^{t_i}x.$$

Thus

$$\begin{aligned}&|\textstyle\sum E([t_i,t_{i+1}])\,T((I+X_\pi)-(I+\tilde{X}^-))\,P^{t_i}x|^2\\&\quad=\textstyle\sum|E([t_i,t_{i+1}])\,T|^2\,|(X_\pi-\tilde{X}^-)\,P^{t_i}x|^2\\&\quad\leqslant |T|_\mu^2\,|X_\pi-\tilde{X}^-|^2\,|x|^2.\end{aligned}$$

Since $S'_\pi(T(I+X_\pi))\to X^+$, it follows then that $X^+=\mathscr{F}_\mu(T(I+\tilde{X}^-))\,G^+$. Hence $X^+\in L^\mu$ and is causal. Returning now to (6), we see that

$$(I-T)=(I-X^+)(I+\tilde{X}^-)^{-1},$$

where the inverse exists by Corollary 3.4. Defining $I-X^-=(I+\tilde{X}^-)^{-1}$, it follows that $X^-\in L^{*\mu}$ (also from Corollary 3.4). Therefore we have the identity

$$-T=-X^+-X^-+X^+X^-,$$

and $X^+=T-X^-+X^+X^-\in L^{*\mu}$ since $L^{*\mu}$ is a left ideal in $B(H)$. Similarly we can show that $X^-\in L^\mu$. Thus X^+, $X^-\in L^\mu\cap L^{*\mu}$. Uniqueness of the factorization follows from the argument given in [1].

Remark. By interchanging P^t with P_t in the proof of the theorem, the alternate factorization $(I-T)=(I+X^-)(I+X^+)$ can also be obtained.

COROLLARY 3.6. *Suppose $T\in L^\mu$ with $\mu\ll\lambda$. Then $(I+TT^*)$ admits the special factorization*

$$(I+TT^*)=(I+V)(I+V^*), \tag{7}$$

where $V\in L^\mu\cap L^{*\mu}$ is causal.

Proof. $T\in L^\mu$ implies that $T^*\in L^{*\mu}$, so that $TT^*\in L^\mu\cap L^{*\mu}$. Since $TT^*\geqslant 0$ and $P^tTT^*P^t\geqslant 0$, the hypotheses of Theorem 3.5 are satisfied. Equation (7) then follows from the uniqueness of the factorization. ∎

Using the preceding remark it also follows that if $T^* \in L^u$, then $(I + T^*T)$ admits the special factorization

$$(I + T^*T) = (I + V^*)(I + V), \tag{8}$$

where $V \in L^u \cap L^{*u}$ is causal.

4. A State Space Application

Let (H_1, E_1) and (H_2, E_2) be two Hilbert resolution spaces and assume $T \in B(H_1, H_2)$ is causal. We think of T as the input/output map of a dynamical system. Now let (H_3, E_3) be another Hilbert resolution space and assume

$$T = CS,$$

where $S \in B(H_1, H_3)$ is causal and $C \in B(H_3, H_2)$ is memoryless. In this section we are concerned with the following question: Under what conditions is it reasonable to assert that H_3 represents the *state space* of the system T? This question has received considerable attention in the resolution space setting by Saeks, Feintuch, Schumitzky, and others (cf. [7–9]).

It is fair to say at this point in time, that the answer to this question depends on the particular application in mind. We will give here a workable definition which is satisfied by all familiar examples and is sufficient for the later applications to problems of optimal "state" feedback control and optimal "state" estimation.

Assume now $S \in L^u_{13}$, then S naturally induces a map $\hat{S} \in B(H_1, H^u_3)$ by the definition

$$x = \hat{S}u,$$

where $u \in H_1$ and $x \in \mathscr{H}^u_3$ is such that

$$x(t) = P_t S P^t u.$$

DEFINITION 4.1. Given the above notations and assumptions, we say $T = CS$ is an *admissible state space decomposition* of T if there exists a memoryless map $M \in B(H_3, \mathscr{H}^u_3)$ such that

$$\hat{S} = MS. \tag{9}$$

In this case, for input $u \in H_1$, we will say that $x(t) = P_t S P^t u$ is the *state* of T at time t, $x = \hat{S}u$ is the *state trajectory* of T, and H_3 is the *state space* of T.

The following familiar example shows that this definition is not unreasonable.

EXAMPLE 4.2. Let H_0 denote a separable Hilbert space and take $H_1 = H_2 = H_3 = H = L^2([0, 1], H_0)$ with the truncation resolution of the identity (cf. Example 1.4). Let B, $C \in B(H_0)$ and let $S(t)$ be a strongly continuous semigroup on H_0. Define the maps T, $S \in B(H)$ by

$$Tu : t \to C \int_0^t S(t-\sigma)\, Bu(\sigma)\, d\sigma, \quad Su : t \to \int_0^t S(t-\sigma)\, Bu(\sigma)\, d\sigma,$$

so that

$$T = CS. \tag{10}$$

It is clear that T and S are causal and C is memoryless and from Example 1.4 it follows that $S < \lambda$.

Finally, define the map $M : H \to \mathscr{H}^u$ by

$$\begin{aligned} (Mz)(t) : \tau &\to CS(\tau - t)\, z(t), && t \leqslant \tau, \\ &\to 0, && t > \tau. \end{aligned}$$

It is easily verified that M is bounded and memoryless and that $\hat{S} = MS$. Thus (10) is an admissible state decomposition.

The map T in the above example is not necessarily Hilbert–Schmidt if H_0 is infinite dimensional. On the other hand, it will be shown in [6] that every Hilbert–Schmidt map T admits an admissible state decomposition.

We can think of the representation $T = CS$ as decomposing T into its *dynamic* part S and its *static* part C (uniqueness not assumed). Then an important question is: When does another map V have the same dynamics as T? The power of the formalism developed in the preceding sections is shown by the following answer to this question.

THEOREM 4.3. *Let $T = CS$ be an admissible state decomposition of T, and let $V \in B(H_1, H_4)$ be a causal map with $V \in L^u_{34}$, where (H_4, E_4) is a Hilbert resolution space. Assume there exists an anticausal map $A \in L^u_{34}$ such that $AS \in L^u_{14}$* and

$$V = [AS]_C.$$

Then there exists a memoryless map $F \in B(H_3, H_4)$ such that

$$V = FS.$$

Further, $F = \mathscr{F}_u(A)\, M$, where M is given by Eq. (9).

Proof. Suppose there exists an anticausal map $A \in L_{34}^{\mu}$ such that $AS \in L_{14}^{\mu}$ and $V = [AS]_C$. Then by definition $V = \mathscr{F}_{\mu}(AS)\,G^{+}$. It can be shown that the anticausality of A implies the identity $V = \mathscr{F}_{\mu}(A)\,\hat{S}$. Equation (9) then gives $V = \mathscr{F}_{\mu}(A)\,MS$. Since M and $\mathscr{F}_{\mu}(A)$ are memoryless and bounded, so is $F = \mathscr{F}_{\mu}(A)\,M$. Thus $V = FS$. ∎

Now let $T \in L_{12}^{*\mu}$ with $\mu \ll \lambda$. By Corollary 3.6, $(I + T^*T)$ admits the special factorization

$$(I + T^*T) = (I + V^*)(I + V) \tag{11}$$

with V, $V^* \in L_{11}^{\mu} \cap L_{11}^{*\mu}$. The following corollary to Theorem 4.3 has important applications to problems in control and estimation theory.

Corollary 4.4. *Let T, V be defined as in Eq.* (11). *Then V has the same dynamics as T. That is, there exists a memoryless map $F \in B(H_3, H_1)$ such that $V = FS$. Further, $F = \mathscr{F}_{\mu}((I + W^*)\,T^*C)\,M$, where $W^* = (I + V^*)^{-1} - I$.*

Proof. By Eq. (11), $T^*T = V + V^* + V^*V = (I + V^*)\,V + V^*$. Let $W^* = (I + V^*)^{-1} - I$. Then

$$V = (I + W^*)\,T^*T - W^*.$$

Now V, $W^* \in L^{\mu}$ are respectively causal and anticausal. Thus by Proposition 2.4.

$$V = [(I + W^*)\,T^*T]_C = [(I + W^*)\,T^*CS]_C = [AS]_C,$$

where $A = (I + W^*)\,T^*C$. Since $T^* \in L_{12}^{\mu}$, $W^* \in L_{11}^{\mu}$ it follows that $A \in L_{31}^{\mu}$, $AS \in L_{11}^{\mu}$. Theorem 4.3 then gives the desired result. ∎

5. Linear Regulator–Quadratic Cost Problem

As an application of the concept of an admissible state decomposition, we show in this section that the optimal control law for the linear regulator–quadratic cost problem can be represented in state feedback form. The development given here follows the algebraic formalism found in [9].

Let (H_i, E_i), $i = 1, 2, 3$, be three Hilbert resolution spaces and let $T \in B(H_1, H_2)$, $S \in B(H_1, H_3)$ be causal and $C \in B(H_3, H_2)$ be memoryless. Assume $T = CS$ is an admissible state decomposition for T, and that $T \in L_{12}^{*\mu}$ and $S \in L_{13}^{\mu}$ with $\mu \ll \lambda$.

Fix $\tau \in (a, b)$ and consider the dynamical system

$$P_\tau y = P_\tau T(P_\tau u + P^\tau w), \qquad u, w \in H_1, y \in H_3, \tag{12a}$$

$$P_\tau x = P_\tau S(P_\tau u + P^\tau w), \qquad x \in H_3. \tag{12b}$$

In Eqs. (12), we think of $P_\tau u$ as the control beginning at time τ corresponding to the input $P^\tau w$ which ends at time τ and $P_\tau y$ and $P_\tau x$ are the corresponding output and state.

Let $\hat{u}$ be the unique minimizer of the quadratic cost

$$J(u) = |P_\tau y|^2 + |P_\tau u|^2$$

subject to the constraint (12).

We call the above optimization problem the (abstract) linear regular—quadratic cost problem. Our goal is to represent the optimal control in state feedback from

THEOREM 5.1. *Given the above notations and assumptions, there exists a memoryless map $F \in B(H_3, H_1)$ such that the optimal control $\hat{u}$ admits the representation*

$$P_\tau \hat{u} = -F P_\tau \hat{x},$$

where

$$P_\tau \hat{x} = P_\tau S(P_\tau \hat{u} + P^\tau w)$$

is the corresponding optimal state.

Proof. Using the projection theorem, $P_\tau \hat{u}$ is shown to satisfy (see Porter [10])

$$P_\tau(I + T^*T) P_\tau \hat{u} = -P_\tau T^*TP^\tau w. \tag{13}$$

However, the special factorization (11) implies

$$T^*T = V^* + V + V^*V = (I + V^*) V + V^*.$$

By the anticausality of V^*, we have

$$P_\tau V^* P^\tau w = P_\tau V^* P_\tau P^\tau w = 0,$$

so that (13) becomes

$$P_\tau(I + V^*)(I + V) P_\tau \hat{u} = -P_\tau(I + V^*) VP^\tau w.$$

By the invertibility of $P_\tau(I + V^*)$ on $P_\tau H_1$ we have

$$P_\tau(I + V)\, P_\tau \hat{u} = -P_\tau V P^\tau w,$$

or equivalently

$$P_\tau \hat{u} = -P_\tau V(P_\tau \hat{u} + P^\tau w).$$

Now applying Corollary 4.4, we have $V = FS$, where $F \in B(H_3, H_1)$ is memoryless. Thus

$$P_\tau \hat{u} = -P_\tau FS(P_\tau \hat{u} + P^\tau w) = -FP_\tau \hat{x}. \quad \blacksquare$$

In a forthcoming paper [6], we will apply the same formalism to the problems of optimal stochastic control and estimation as outlined in [3].

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